EXTRA STRUCTURE AND THE UNIVERSAL CONSTRUCTION FOR THE WITTEN-RESHETIKHIN-TURAEV TQFT

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ABSTRACT. A TQFT is a functor from a cobordism category to the category of vector spaces, satisfying certain properties. An important property is that the vector spaces should be finite dimensional. For the WRT TQFT, the relevant $2 + 1$-cobordism category is built from manifolds which are equipped with an extra structure such as a p_1-structure, or an extended manifold structure. We perform the universal construction of [BHMV95] on a cobordism category without this extra structure and show that the resulting quantization functor assigns an infinite dimensional vector space to the torus.

1. INTRODUCTION

A TQFT in dimension $2 + 1$ is a covariant functor (V, Z) from some $(2 + 1)$-cobordism category $\mathcal{C}$ to the category of finite dimensional complex vector spaces which assigns to the empty object the vector space $\mathbb{C}$. Other properties are usually required for a TQFT, and other ground rings are sometimes allowed. But for our purposes, this will do.

Recall an object Σ in $\mathcal{C}$ is a closed oriented surface with possibly some specified extra structure, and a morphism C from Σ_1 to Σ_2 is an equivalence class of cobordisms from Σ_1 to Σ_2. Such a cobordism can be loosely viewed as a compact oriented 3-manifold (again possibly with some appropriate extra structure) with a boundary decomposed into an incoming surface $-\Sigma_1$ and an outgoing surface Σ_2. Two cobordisms are considered equivalent if there is a orientation-preserving (extra structure preserving) diffeomorphism between them which restricts to the identity on the boundary. Then (V, Z) assigns a vector space $V(\Sigma)$ to an object Σ, and a linear map $Z_C : V(\Sigma_1) \to V(\Sigma_2)$ to a morphism $C : \Sigma_1 \to \Sigma_2$.

The WRT-invariant is a 3-manifold invariant which was first described by Witten in [Wit89] and then rigorously defined by Reshetikhin and Turaev with quantum groups in [RT91]. The approach to this invariant that we will use was developed by Blanchet, Habegger, Masbaum, and Vogel in [BHMV92] with skein theory and then used by them to construct [BHMV95] a TQFT on a 2+1 cobordism category where the objects and morphisms are equipped with p_1-structures. The question that we consider in this paper is whether this construction based on the WRT-invariant still yields a TQFT when the extra structure is removed from the cobordism category. The answer is no, as the resulting vector space associated to the torus has infinite dimension. See Theorem 1. To be more precise: we follow this construction after

assigning to each closed 3-manifold the invariant of this 3-manifold equipped with a certain choice of extra structure. Our choice, which seems to us to be the most natural, is described in the next paragraph.

For each integer $p \geq 5$, consider the complex valued invariant, $\langle \ \rangle_p$ of closed oriented 3-manifolds equipped with a p_1-structure defined in [BHMV95]. Here we must choose a primitive $2p$th root of unity $A \in \mathbb{C}$, and scalar $\kappa \in \mathbb{C}$ with $\kappa^6 = A^{-6-p(p+1)/2}$. One may remove the dependence on this extra structure by defining $\langle M \rangle'_p = \langle \check{M} \rangle_p$ where $\check{M}$ is M equipped with a p_1-structure with σ-invariant zero. See [BHMV95, Appendix B] for the definition of the σ-invariant. If one uses extended manifold structures in lieu of p_1-structures as in [Wal91, Tur94, GM12], one would instead choose $\check{M}$ to have weight zero.

If M is obtained by surgery to S^3 along a framed link L, then

$$\langle M \rangle'_p = \eta \mu^{-\sigma(L)} L(\omega_p).$$

Here we let $\mu = \kappa^3$ and $\sigma(L)$ stands for the signature of the linking matrix of framed link L. Also ω_p is the skein specified in [BHMV95, p.898], η is the scalar as given in [BHMV95, p.897] and $L(\omega_p)$ is the Kauffman bracket of the cabling of L by ω_p. One can easily extend this definition to the disconnected case by letting $\langle M_1 \sqcup M_2 \rangle'_p = \langle M_1 \rangle'_p \langle M_2 \rangle'_p$.

A quantization functor is a covariant functor (V, Z) from $\mathcal{C}$ to a category of (not necessarily finite dimensional) complex vector spaces. Like a TQFT, it should assign to the empty object the vector space $\mathbb{C}$. A certain naturally defined Hermitian form on $V(\Sigma)$ must also be non-degenerate. One has that $\langle \ \rangle'_p$ is multiplicative and involutive. So we can perform the universal construction described in [BHMV95, Prop. 1.1] to construct a quantization functor from the ordinary $(2+1)$-cobordism category (without any extra structure), which we will denote by $\mathcal{C}'$, to the category of complex vector spaces

This is how the universal construction goes (when applied to $\mathcal{C}'$ and $\langle \ \rangle'_p$): Given an object Σ in $\mathcal{C}'$, denote $\mathcal{V}'_p(\Sigma)$ as the vector space spanned by all compact oriented 3-manifolds with boundary Σ (or equivalently all cobordisms $\{M : \emptyset \to \Sigma\}$). There is a hermitian form $\langle \ , \ \rangle'_\Sigma$ on $\mathcal{V}'_p(\Sigma)$; This is specified on generators by

$$\langle M, N \rangle'_\Sigma = \langle M \cup_\Sigma -N \rangle'_p$$

and then extended sesquilinearly. Let $\mathrm{rad}\langle \ , \ \rangle'_\Sigma$ denote the radical of the hermitian form $\langle \ , \ \rangle'_\Sigma$. Define $V'_p(\Sigma)$ to be $\mathcal{V}'_p(\Sigma)/\mathrm{rad}\langle \ , \ \rangle'_\Sigma$. Given a morphism $C : \Sigma_1 \to \Sigma_2$, define $\mathcal{Z}'_{p,C} : \mathcal{V}'_p(\Sigma_1) \to \mathcal{V}'_p(\Sigma_2)$ by assigning $C \cup_{\Sigma_1} N$ to any $N \in \mathcal{V}'_p(\Sigma_1)$ and extending linearly. Note that $\mathcal{Z}'_{p,C}$ send $\mathrm{rad}\langle \ , \ \rangle'_{\Sigma_1}$ into $\mathrm{rad}\langle \ , \ \rangle'_{\Sigma_2}$. So it induces a linear map $Z'_{p,C} : V'_p(\Sigma_1) \to V'_p(\Sigma_2)$. Then the quantization functor (V'_p, Z'_p) is the rule assigning $V'_p(\Sigma)$ to Σ and $Z'_{p,C}$ to C.

Let $\mathcal{S}$ be a standard unknotted solid torus in 3-space, and let T^2 denote the boundary of $\mathcal{S}$. Let w_i denote the 3-manifold obtained by doing surgery to $\mathcal{S}$ along i parallel copies of the core of $\mathcal{S}$ with framing $+1$. Let z_j denote the 3-manifold obtained by doing surgery along the core of $\mathcal{S}$ with framing j. We have the following theorem which will be proved in the next section.

Theorem 1. For all $p \geq 5$, $V'_p(T^2)$ is infinite-dimensional. An infinite set of linearly independent elements in $V'_p(T^2)$ can be given by either $\{w_{2pk}\}$ or $\{z_{2pl}\}$, where k and l vary through the positive integers. Hence V'_p is not a TQFT.

If (V, Z) is a quantization functor resulting from the universal construction, then [BHMV95, p.886] there is a natural map

$$t^V_{\Sigma_1, \Sigma_2} : V(\Sigma_1) \otimes V(\Sigma_2) \to V(\Sigma_1 \sqcup \Sigma_2).$$

It is easy to see that this map must be injective. Quinn [Qui95, Prop. 7.2] gave an argument that shows that the finite-dimensionality of $V(\Sigma)$ is implied by the functoriality of V applied to a "snake-shaped" composition of cobordisms built from copies of $\Sigma \times I$ and the assumption that $t^V_{\Sigma, \Sigma}$ is an isomorphism. See also Kock [Koc03, Corollary 1.2.28]. This argument shows:

Corollary 1. The natural map $t^{V'_p}_{T^2, T^2}$ is not surjective.

2. Proof of Theorem

We first construct a bilinear form $B_p(\ ,\ )$ on $V'_p(T^2)$, then write out the $n \times n$ truncated matrix associated to $B_p(\ ,\ )$ with respect to $\{w_{2pk}\}$ and $\{z_{2pl}\}$ as k, and l range from 1 to n. We will show there are infinitely many integers n such that the truncated matrix of size $n \times n$ is nonsingular. Hence $\{w_{2pk}\}$ and $\{z_{2pl}\}$ are linearly independent and $V'_p(T^2)$ is infinite-dimensional.

2.1. **Bilinear form on $V'_p(T^2)$.** Every closed orientable connected 3-manifold can be obtained by doing Dehn surgery in S^3, see [Lic97]. This result can be used to show the following well-known related fact: every connected orientable 3-manifold with boundary T^2 can be obtained by doing surgery along some framed link L in the solid torus $\mathcal{S}$. We denote the result of this surgery by $\mathcal{S}(L)$.

According to the universal construction, elements in $V'_p(T^2)$ are represented by linear combinations of connected manifolds with boundary T^2. Given two elements w and z in $V'_p(T^2)$ represented by $\mathcal{S}(L_w)$ and $\mathcal{S}(L_z)$, we glue together $\mathcal{S}(L_w)$ to $\mathcal{S}(L_z)$ by the map on their boundary tori which switches the meridian and the longtitude (note this map is orientation reversing). The resulting manifold is obtained by performing surgery on the 3-sphere along a framed link $(L_w, L_z)_{\text{Hopf}}$ obtained by cabling the Hopf link with L_w on one component and L_z on the other component. We define

$$B_p(w, z) = \langle S^3((L_w, L_z)_{\text{Hopf}}) \rangle'_p \ .$$

This extends to a well-defined symmetric bilinear form $B_p : V'_p(T^2) \times V'_p(T^2) \to \mathbb{C}$, as elements in $\text{rad}\langle\ ,\ \rangle'_{T^2}$ will pair with any other element to give zero.

2.2. **Truncated square matrices.** Note that $B_p(w_{2pk}, z_{2pl})$ is Kauffman bracket of the 3-manifold obtained by doing sugery to S^3 along the framed link pictured in Figure 1. Blowing down [Kir78] the $2pk$ unknotted components with framing $+1$, one by one, and reducing the framing of the single component linked with these, we get an unknot L' with framing $2p(l - k)$. Doing this surgery gives us the lens space $L(2p(l - k), 1)$.

Thus $B(w_{2pk}, z_{2pl}) = \langle L(2p(l - k), 1) \rangle'_p$. We let s abbreviate $l - k$ to simplify our expressions. To compute $\langle L(2ps, 1) \rangle'_p$, we need to compute $U(t^{2ps}\omega_p)$. Here t is the linear map from Kauffman skein module $K(S^1 \times D^2)$ to itself induced by a positive twist and U stands for an unknot with framing zero. According to [BHMV92], $t^{2ps}e_i = u_k^{2ps}e_i$, where $u_k = (-A)^{k(k+2)}$ and the e_i are certain generators for $K(S^1 \times D^2)$.

2pk copies

2pl crossings
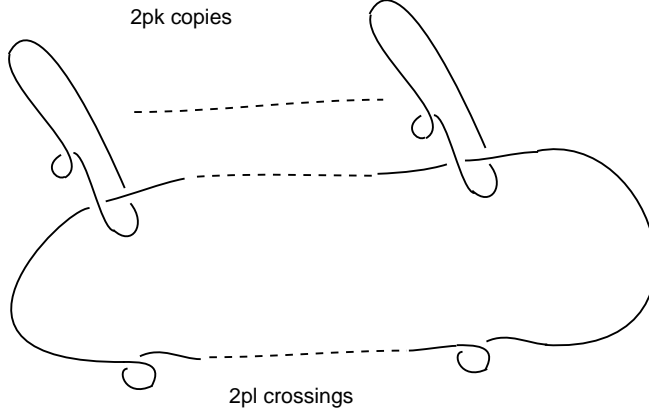
FIGURE 1. Framed link $(L_{w_{2pk}}, L_{z_{2pl}})_{\text{Hopf}}$

As A is a 2p-th root of unity, and $u_k^{2ps} = 1$, it follows that t^{2ps} is the identity on $K(S^1 \times D^2)$. We obtain

$$\langle L(2ps, 1)\rangle'_p = \eta U(t^{2ps}\omega_p)\mu^{-\sigma(L')} = \eta U(\omega_p)\mu^{-\sigma(L')} = \mu^{-\sigma(L')} \ .$$

This last equation holds as $\eta U(\omega_p) = 1$ [BHMV95, p.897]. Alternatively,

$$\eta U(\omega_p) = \langle S^1 \times S^2\rangle'_p = \dim(V_p(S^2)) = 1.$$

Consider the matrix with entries $B(w_{2pk}, z_{2pl})$ as k, l range over the integers from 1 to n (actually any set of n integers in increasing order will do). We will call this a truncated matrix of size n. Then each entry in a truncated matrix just depends on the signature of the linking matrix of L'. Hence every truncated matrix has the form:

$$(*) \qquad \begin{bmatrix} 1 & \mu & \mu & \dots & \mu \\ \mu^{-1} & 1 & \mu & \dots & \mu \\ \mu^{-1} & \mu^{-1} & 1 & \dots & \mu \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mu^{-1} & \mu^{-1} & \mu^{-1} & \dots & 1 \end{bmatrix} \ .$$

2.3. **Determinants of the truncated matrices.** We will not try to show that the truncated matrix of every size is nonsingular. In fact, the truncated 2×2 matrix has determinant zero. Instead, we show that two truncated matrices of consecutive sizes can not be both singular. Therefore infinitely many nonsingular truncated matrices exist.

Consider the $m \times m$ matrix $\mathcal{B}(a, m)$:

$$\begin{bmatrix} a & a-(1-\mu) & a-(1-\mu) & \dots & a-(1-\mu) \\ \mu^{-1} & 1 & \mu & \dots & \mu \\ \mu^{-1} & \mu^{-1} & 1 & \dots & \mu \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mu^{-1} & \mu^{-1} & \mu^{-1} & \dots & 1 \end{bmatrix} \ .$$

Note that $\mu \neq 1$ for $p \geq 5$. We can apply the following elementary row and column operations to $\mathcal{B}(a, m)$: subtract the second column from the first column, add $-\mu^{-1}$

times the first row to the second row, then clearing all but the first entry in the first row, and obtain $(1 - \mu) \oplus \mathcal{B}(f(a), m - 1)$. Here $f(a) = \mu^{-1}(1 - a)$. Applying same operations to the $\mathcal{B}(f(a), m - 1)$ part of $(1 - \mu) \oplus \mathcal{B}(f(a), m - 1)$, we obtain $(1 - \mu)I_2 \oplus \mathcal{B}(f^2(a), m - 2)$. Repeating this q times for some $q \leq m - 1$, we see that $(1 - \mu)I_q \oplus \mathcal{B}(f^q(a), m - q)$ is equivalent to $\mathcal{B}(a, m)$. Here we say two matrices are equivalent if they are related by a sequence of determinant preserving elementary row and column operations.

Note that $\mathcal{B}(1, n)$ is exactly the matrix (*) of size n. Following from the above argument, it is clear that $\mathcal{B}(1, n)$ is equivalent to $(1 - \mu)I_{n-1} \oplus \mathcal{B}(f^{n-1}(1), 1)$ and that $\mathcal{B}(1, n+1)$ is equivalent to $(1 - \mu)I_{n-1} \oplus \mathcal{B}(f^{n-1}(1), 2)$. So both $\det \mathcal{B}(1, n)$ and $\det \mathcal{B}(1, n+1)$ can be written in terms of $f^{n-1}(1)$ as $\det \mathcal{B}(1, n) = (1 - \mu)^{n-1}f^{n-1}(1)$ and $\det \mathcal{B}(1, n+1) = (1 - \mu)^{n-1}(f^{n-1}(1)(1 - \mu^{-1}) + (\mu^{-1} - 1))$. Therefore we have

$$\det \mathcal{B}(1, n+1) = \det \mathcal{B}(1, n)(1 - \mu^{-1}) + (1 - \mu)^{n-1}(\mu^{-1} - 1) \ .$$

As a result, as long as $\mu \neq 1$, we can not have two consecutive singular truncated matrices since the term $(1 - \mu)^{n-1}(\mu^{-1} - 1)$ is non-zero. This completes the proof of Theorem 1.

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